ON THE STABILITY OF MOTION IN A CERTAIN CRITICAL CASE

(OB USTOICHIVOSTI DVIZHENIIA V ODNOM KRITICHESKOM SLUCHAE)

PMM Vol.30, № 4, 1966, pp.779-782

NGO VAN VYONG
(Hanoi, Viet-Nam)

(Received October 30, 1965)

We shall consider the stability of motion described by a system of differential equations of perturbed motion of the form

$$x_i^{\cdot} = y_i + X_i^*, \quad y_i^{\cdot} = Y_i^*, \quad \zeta_s^{\cdot} = \sum_{k=1}^{n} p_{sk}\zeta_k + Z_s^* \qquad \begin{pmatrix} i=1, \ldots, m \\ s=1, \ldots, n \end{pmatrix} \tag{0.1}$$

Here X_i^*, Y_i^*, Z_s^* are holomorphic functions which contain no terms of lower than the second order in $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; $\zeta_1, \ldots, \zeta_n$. All the roots of Equation $|p_{sk} - \delta_{sk}\lambda| = 0$ have negative real parts different from zero.

Paper [1] contains a discussion of a system of the form (0.1) under the condition

$$Y_i^* = \sum_{k=1}^{m} a_{ik}y_k^2 + \sum_{k=1}^{m} P_{ik}(\zeta_1, \ldots, \zeta_n)\, y_k + Q_i(\zeta_1, \ldots, \zeta_n) + \sum_{\sigma=1}^{n} \zeta_\sigma \varphi_{i\sigma}(x_1, \ldots, x_m) + R_i(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ \zeta_1, \ldots, \zeta_n) \tag{0.2}$$

$$Z_s^* = \sum_{\sigma=1}^{n} \zeta_\sigma \omega_{s\sigma}(x_1, \ldots, x_m) + R_s'(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ \zeta_1, \ldots, \zeta_n)$$

where a_{ik} are constants; $\varphi_{i\sigma}$, $\omega_{s\sigma}$ are holomorphic functions which vanish for $x_1 = \ldots = x_m = 0$; P_{ik} are linear and Q_i are quadratic forms in $\zeta_1, \ldots, \zeta_n$; R_i, R_s' are holomorphic functions in $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; $\zeta_1, \ldots, \zeta_n$ which contain no terms of lower than the third order in these variables. In paper [1] an attempt is made to prove the instability of unperturbed motion with values of Y_i^*, Z_s^* which satisfy condition (0.2).

Despite the highly specific form of the functions Y_i^*, Z_s^* considered in [1], the function V proposed by its authors is not a Chetaev function unless additional conditions are imposed on Y_i^* and Z_s^*. In fact, the expression for S ([1], (2.8)) contains, for example, the sums

$$\sum_{k=1}^{m} \left[1 + \left(1 - \sum_{k=1}^{m} a_{ik}\right) x_k\right] R_i, \qquad \sum_{k=1}^{m} \sum_{s=1}^{n} \psi_{ks}(x_1, \ldots, x_m)\, R_s'$$

which include terms of the form $x_i^{\delta i}$, $y_i x_i^{\nu i}$, $\zeta_s x_i^{\mu i}$ $(\delta_i, \mu_i, \nu_i \geqslant 2)$. It is clear that in the presence of such terms the derivative dV/dt can take on values with either sign when $V > 0$. Hence, in choosing the function V in accordance with (2.5) in [1], the following additional conditions on the functions Y_i^* and Z_s^* must be imposed:

1) For $y_1 = \ldots = y_m = \zeta_1 = \ldots = \zeta_n = 0$, all $Y_i^* \equiv 0$, $Z_s^* \equiv 0$.

2) None of the quantities R_i and R_s' contain terms of lower than the second order in $y_1 \ldots y_m$, $\zeta_1 \ldots \zeta_n$.

1. Let us investigate system (0.1) assuming that X_i^*, Z_s^* vanish for $y_1 = \ldots = y_m = \zeta_1 = \ldots = \zeta_n = 0$. This assumption does not restrict the generality of the problem considered in [1]. We transform system (0.1), setting

$$\zeta_s = z_s + u_s(x_1, \ldots, x_m;\ y_1, \ldots, y_m) \qquad (s = 1, \ldots, n) \tag{1.1}$$

where $u_s(x_1, \ldots, x_m;\ y_1, \ldots, y_m)$ are the roots of Equations

$$p_{s1}u_1 + \ldots + p_{sn}u_n + Z_s^*(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ u_1, \ldots, u_n) = 0$$

As a result we have

$$x_i' = y_i + X_i, \quad y_i' = Y_i, \quad z_s' = \sum_{k=1}^{n} p_{sk} z_k + Z_s \qquad \begin{pmatrix} i=1, \ldots, m \\ s=1, \ldots, n \end{pmatrix} \tag{1.2}$$

Here X_i, Y_i are the values of the functions X_i^*, Y_i^* for $\zeta_s = z_s + u_s$, and

$$Z_s = Z_s^*(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ z_1 + u_1, \ldots, z_n + u_n) -$$

$$- Z_s^*(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ u_1, \ldots, u_n) - \sum_{k=1}^{m} (y_k + X_k) \frac{\partial u_s}{\partial x_k} - \sum_{k=1}^{m} Y_k \frac{\partial u_s}{\partial y_k} \tag{1.3}$$

We note that (a) the functions $X_i = 0$ for $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$; (b) the functions Y_i and Z_s vanish identically for $y_1 = \ldots = y_m = 0$, $z_1 = \ldots = z_n = 0$ provided all the functions Y_i^* vanish identically for $y_1 = \ldots = y_m = \zeta_1 = \ldots = \zeta_n = 0$; (c) the functions Z_s do not contain terms with first powers of $y_1, \ldots, y_m$ for $z_1 = \ldots = z_n = 0$ provided all the Y_i^* vanish for $y_1 = \ldots = y_m = \zeta_1 = \ldots = \zeta_n = 0$ and Y_i do not contain terms with first powers of $y_1, \ldots, y_m$ for $z_1 = \ldots = z_n = 0$.

Let us suppose that $Y_i = 0$ $(i = 1, \ldots, m)$ for $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$, and, in addition, that for $z_1 = \ldots = z_n = 0$ the functions Y_i do not contain linear terms in $y_1, \ldots, y_m$. We shall prove that the unperturbed motion is unstable.

We take the Chetaev function in the form

$$V = \sum_{i=1}^{m} x_i y_i + \sum_{s=1}^{n} z_s \vartheta_s(x_1, \ldots, x_m) + W(z_1, \ldots, z_n) \tag{1.4}$$

Here $W(z_1, \ldots, z_n)$ is a negative definite quadratic form which satisfies Equation

$$\sum_{s=1}^{n} \frac{\partial W}{\partial z_s} (p_{s1}z_1 + \cdots + p_{sn} z_n) = \sum_{s=1}^{n} z_s^2$$

and ϑ_s are holomorphic functions of $x_1, \ldots, x_m$ which vanish for $x_1 = \ldots = x_m = 0$ and satisfy Equations

$$\sum_{i=1}^{m} x_i F_{ik} + \sum_{s=1}^{n} \vartheta_s (p_{sk} + Q_{sk}) = 0 \qquad (k = 1, \ldots, n)$$

$$F_{ik} = \frac{\partial Y_i}{\partial z_k}\bigg|_{z=y=0}, \qquad Q_{sk} = \frac{\partial Z_s}{\partial z_k}\bigg|_{z=y=0}$$

With the functions ϑ_s, chosen in this way and by virtue of the structure of the right-hand sides of the system (1.2), the derivative V' can be written as

$$V' = \sum_{i=1}^{m} y_i^2 + \sum_{s=1}^{n} z_s^2 + \sum_{i=1}^{m} \sum_{k=1}^{m} y_i y_k \varphi_{ik} + \sum_{j=1}^{n} \sum_{\sigma=1}^{n} z_j z_\sigma \psi_{j\sigma} + \sum_{i=1}^{m} \sum_{j=1}^{n} y_i z_j f_{ij}$$

Here the functions φ_{ik}, $\psi_{j\sigma}$, f_{ij} vanish for

$$x_1 = \ldots = x_m = y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$$

Let us consider the domain $V > 0$. It is clear that in this domain the derivative V' is a positive quantity and vanishes only on the boundary of the domain $V > 0$, where $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$. Hence, the unperturbed motion is unstable [2].

Note. It is easy to show that in the case of motion just analyzed, $x_i = c_i$ $(i = 1, \ldots, m)$, $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$ are also unstable for sufficiently small values of the constants c_i.

2. Let us consider the case where $Y_i \not\equiv 0$ for $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$. Let us suppose that the functions Y_i for $z_1 = \ldots = z_n = 0$ do not contain linear terms in $y_1, \ldots, y_m$. In system (1.2) let (2.1)

$$Y_i = \sum_{k=1}^{m} g_{ik} x_k^{r_{ik}} + \sum_{k=1}^{m} x_k^{r_{ik}} f_{ik}(x_1, \ldots, x_m) + \sum_{k=1}^{m} a_{ik} y_k^2 + \sum_{\sigma=1}^{n} z_\sigma \varphi_{i\sigma}(x_1, \ldots, x_m) +$$
$$+ \sum_{k=1}^{m} P_{ik}(z_1, \ldots, z_n) y_k + Q_i(z_1, \ldots, z_n) + R_i(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ z_1, \ldots, z_n)$$
$$Z_s = \sum_{\sigma=1}^{n} z_\sigma \omega_{s\sigma}(x_1, \ldots, x_m) + R_s'(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ z_1, \ldots, z_n) \quad \begin{pmatrix} i=1, \ldots, m \\ s=1, \ldots, n \end{pmatrix}$$

Here a_{ik}, g_{ik} are constants; $f_{ik}, \varphi_{i\sigma}, \omega_{s\sigma}$ are holomorphic functions of $x_1, \ldots, x_m$ which vanish for $x_1 = \ldots = x_m = 0$; P_{ik} are linear and Q_i are quadratic forms in $z_1, \ldots, z_n$; R_i are holomorphic functions of $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; $z_1, \ldots, z_n$ which do not contain terms of lower than the third dimension in these variables and do not include terms of lower than the second order in $y_1, \ldots, y_m$; $z_1, \ldots, z_n$; R_s' are holomorphic functions of $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; $z_1, \ldots, z_n$ which vanish for $x_k = y_k = z_s = 0$ and do not contain terms with first powers of $z_1, \ldots, z_n$ for $y_1 = \ldots = y_m$.

We note that if for $z_1 = \ldots = z_n = 0$ the functions Y_i contain, for example terms in $x_i^{r_{ik}}$, then for $z_1 = \ldots = z_n = 0$ the functions Z_s can contain similar terms whose order relative to x_i is higher by at least one. This property of the functions Z_s follows from Equations (1.3)

We shall show that unperturbed motion is unstable if:

a) the nonlinear functions Y_i, Z_s satisfy conditions (2.1);

b) in each column of the matrix

$$\begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \cdot & \cdots & \cdot \\ r_{m1} & \cdots & r_{mm} \end{pmatrix} \tag{2.2}$$

the smallest numbers $r_{i'k}$ $(k = 1, \ldots, m)$ are even and the corresponding quantities $g_{i'k}$ have same sign.

Let us take a Liapunov function V of the form [1]

$$V = \sum_{k=1}^{m} \left[1 + \left(\sum_{i'} g_{i'k} - \sum_{i=1}^{m} a_{ik}\right) x_k\right] y_k + \sum_{k=1}^{m} \sum_{s=1}^{n} z_s \psi_{ks} +$$
$$+ \sum_{i=1}^{m} \sum_{k=1}^{m} U_{ik}(z_1, \ldots, z_n) y_k + \sum_{k=1}^{m} W_k \tag{2.3}$$

Here ψ_{ks} are functions of the variables $x_1, \ldots, x_m$ which satisfy Equations

$$\sum_{s=1}^{n} (p_{s\sigma} + \omega_{s\sigma}) \psi_{ks} + \left[1 + \left(\sum_{i'} g_{i'k} - \sum_{i=1}^{m} a_{ik}\right) x_k\right] \varphi_{k\sigma} = 0 \qquad \begin{pmatrix} \sigma=1, \ldots, n \\ k=1, \ldots, m \end{pmatrix}$$

The linear and quadratic forms U_{ik}, W_k of the variables $z_1, \ldots, z_n$ can be determined from Equations

$$\sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial z_s}(p_{s1}z_1 + \cdots + p_{sn}z_n) + P_{ik} = -\sum_{s=1}^{n} z_s \left(\frac{\partial \psi_{is}}{\partial x_k}\right)_0 \qquad \begin{pmatrix} i = 1, \ldots, m \\ k = 1, \ldots, m \end{pmatrix}$$

$$\sum_{s=1}^{n} \frac{\partial W_k}{\partial z_s}(p_{s1}z_1 + \cdots + p_{sn}z_n) + Q_k = \sum_{i'} \sum_{s=1}^{n} g_{i'k} z_s^2$$

Functions V of the form (2.3) satisfy Liapunov's theorem on instability [3]. Hence, the unperturbed motion is unstable.

The unperturbed motion is also unstable if:

a) the nonlinear functions Y_i, Z_s satisfy conditions (2.1);

b) the diagonal elements r_{kk} of the matrix (2.2) are odd and smaller than the elements of the corresponding column, and if, moreover, the quantities $g_{kk} > 0$.

In this case the Liapunov function V can be taken in the form

$$V = \sum_{k=1}^{m} \left(g_{kk}x_k + \sum_{i=1}^{m} U_{ik}\right) y_k + \sum_{k=1}^{m} \sum_{s=1}^{n} z_s \psi_{ks} + W \qquad (2.4)$$

Here ψ_{ks} are functions of $x_1, \ldots, x_s$ which satisfy Equations

$$\sum_{s=1}^{n} (p_{s\sigma} + \omega_{s\sigma}) \psi_{ks} + g_{kk} x_k \varphi_{k\sigma} = 0 \qquad \begin{pmatrix} \sigma = 1, \ldots, n \\ k = 1, \ldots, m \end{pmatrix}$$

The linear and quadratic forms $U_{ik}W$ can be determined from Equations

$$\sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial z_s}(p_{s1}z_1 + \cdots + p_{sn}z_n) = -\sum_{s=1}^{n} z_s \left(\frac{\partial \psi_{is}}{\partial x_k}\right)_0 \qquad \begin{pmatrix} i = 1, \ldots, m \\ k = 1, \ldots, m \end{pmatrix}$$

$$\sum_{s=1}^{n} \frac{\partial W}{\partial z_s}(p_{s1}z_1 + \cdots + p_{sn}z_n) = \sum_{s=1}^{n} z_s^2$$

Functions V of the form (2.4) satisfy Liapunov's theorem on instability [3]. The unperturbed motion is therefore unstable.

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Translated by A.Y.